THE CHROMATIC NUMBER OF KNESER GRAPHS

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1. The Kneser Graph

Some of the most enlightening proofs in mathematics are those which combine two seemingly unrelated subfields for surprising insights. One great such proof is the proof of Kneser's conjecture, which deals with coloring a special type of graph called the *Kneser graph*. The proof of Kneser's conjecture, which was found by László Lovász, uses the Bursuk-Ulam theorem from topology, and was refined by Joshua Greene. [1]. In this paper, we will present Greene's proof of Kneser's conjecture and prove several related extensions: Dolnikov's theorem and Schrijver's theorem.

Definition 1.1 (Kneser graph)**.** A *Kneser graph* is a graph of the following form, for $n \geq k \geq 1$: its vertices are k-subsets of $[n]$, and vertices corresponding to disjoint subsets are adjacent, i.e. there is an edge between them.

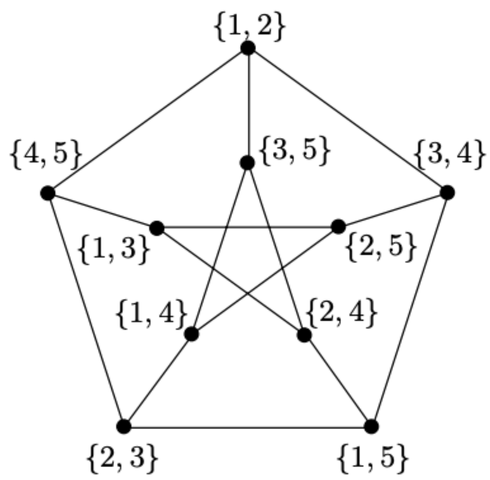

Figure 1. A well-known Kneser graph: $K(5,2)$, or the Petersen graph.

Kneser graphs have some interesting properties that combine graph theory and finite sets; consider, for example, the independence number:

Remark 1.2. The *independence number* of a graph is the maximum size of a set of vertices that are pairwise edgeless; for Kneser graphs, this number is $\binom{n-1}{k-1}$.

We know this from the Erdos-Ko-Rado theorem, which we have covered in class and deals with the number of elements in an intersecting family of k-subsets.

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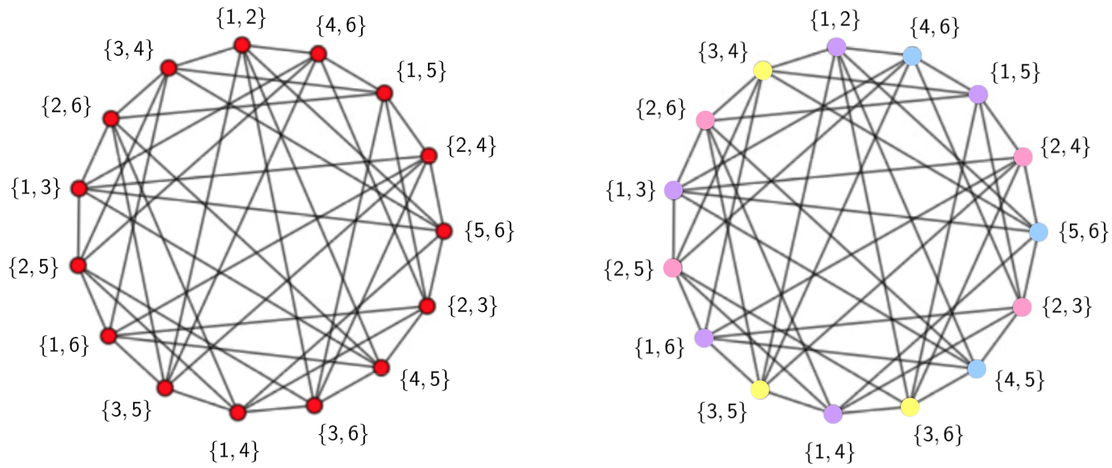


(a) $K(6, 2)$ **(b)** A coloring

Figure 2. The Kneser graph $K(6, 2)$ and its coloring.

Also, Kneser graphs with $k = 1$ are just the complete graphs with n vertices: each subset is a single element from $[n]$, and it is disjoint from all of the other subsets. Therefore, $K(n, 1)$ is a complete graph.

Now, if $n < 2k$, any two k-subsets must intersect, so all vertices of the corresponding Kneser graph are disjoint. Therefore, the only nontrivial Kneser graphs are those where $n \geq 2k$; we will assume this from now on.

Definition 1.3. The chromatic number of a graph G with vertex set V, denoted by $\chi(G)$, is the minimum number of colors needed for a valid coloring of G; that is, the smallest value m such that there is a mapping $V \to [m]$ where adjacent vertices are colored differently.

A coloring is a partition of the vertices into disjoint sets $V_1 \cup V_2 \cup \ldots \cup V_k$ such that each set is edgeless; in terms of Kneser graphs, this is a partition of k-subsets of $[n]$ into disjoint sets $V_1 \cup V_2 \cup \ldots \cup V_k$ such that each V_i is an intersecting family of k-sets, because 2 subsets are connected by an edge in a Kneser graph if they are *disjoint*. Let us write $n = 2k+d$ where $d \geq 0$.

There is an easy coloring of $K(n, k)$ that uses $d+2$ colors. It goes like this. For $1 \leq i \leq d+1$, let V_i be the set of all subsets with i as its smallest element (these are obviously intersecting). All remaining subsets are subsets of $\{d+2, d+3, \ldots 2k+d\}$, which has $2k-1$ elements; therefore, they are intersecting and can be assigned the color $d+2$. So, we have shown that $\chi(K(n, k) \leq d+2$.

Example. Let's color a Kneser graph with this method. We will color $K(6, 2)$, which is shown in Figure 2. We have $n = 6$ and $k = 2$, so $d = 2$ and we will use 4 colors.

Kneser's conjecture states that this number $d+2$, is the right number; ie, that there are no colorings of Kneser graphs with fewer colors.

2. Coloring the Kneser Graph

Conjecture 2.1 (Kneser's conjecture)**.** *We have* $\chi(K(2k+d, k)) = d+2$.

This conjecture was proposed, but not proved, by Kneser when he introduced the concept of Kneser graphs in 1955. A first topological proof was found by László Lovász in 1978, and has been refined in the years since.

We showed in Section 1 that the chromatic number is at most $d+2$. Therefore, we must show that $\chi(K(2k+d,k)) \geq d+2$, or alternatively, that $\chi(K(2k+d,k)) \not< d+2$.

We rephrase the conjecture as an existence question: we must show that if k-subsets of $[2k+d]$ are partitioned into $d+1$ classes $V_1 \cup V_2 \cup \ldots \cup V_{d+1}$, at least one of the classes contains a pair of disjoint subsets, as this would make the coloring invalid.

The insight that lead to the proof of this theorem was its connection to the Borsuk-Ulam theorem, which deals with the d-dimensional sphere S^d in $\mathbb{R}^{d+1}$.

Theorem 2.2 (Borsuk-Ulam theorem)**.** *Given a continuous map $f : S^d \to \mathbb{R}^d$, there is at least one pair of antipodal points $x^*, -x^*$ that are mapped to the same point $f(x^*) = f(-x^*)$.*

We will not prove this theorem, as it is out of the scope of this paper; however, we will restate it slightly to aid in our proof.

Theorem 2.3 (Lyusternik–Shnirel'man theorem)**.** *If S^d is covered by the sets $U_1 \ldots U_{d+1}$, such that the first d sets are either open or closed, then one of U_i's contains a pair of antipodal points x^* and $-x^*$.*

For Kneser's conjecture, we only need the case where the first d sets are open, but we prove the full conjecture regardless.

Proof. Say we have a covering of S^d, as stated; we will assume that none of the U_k's have antipodal points. Define a map $f : S^d \to \mathbb{R}^d$:

$$f(x) = (d(x, U_1), d(x, U_2), \ldots d(x, U_d))$$

where d is the minimum Euclidean distance from x to any point in the set U_i; because distance is a continuous map, f itself is continuous as well. Then, by the Borsuk-Ulam theorem, f contains a pair of antipodal points $x^*, -x^*$ such that $f(x^*) = f(-x^*)$.

We have assumed that U_{d+1} does not have any antipodes, so one or more of $x^*, -x^*$ must be in some U_k. We assume that this is x^* (switching the two antipodes if needed). Then, we have $d(x^*, U_k) = 0$, and because $f(x^*) = f(-x^*)$, $d(-x^*, U_k) = 0$ as well.

If the set U_k is closed, then $d(-x^*, U_k) = 0$ means that $-x^* \in U_k$, which is a contradiction. If U_k is open, then $-x^* \in \overline{U_k}$, the closure, or the smallest closed set containing U_k. $\bar{U}_k \in S^d \setminus (-U_k)$, because this is a closed set containing U_k. But then $-x^*$ lies in $S^d \setminus (-U_k)$, so it cannot lie in $-U_k$, so x^* cannot lie in U_k; this is also a contradiction. □

The original proof of Kneser's conjecture used the Lyusternik-Shnirel'man theorem along with Gale's lemma, which deals with arrangements of n points on S^d such that each hemisphere contains k points. However, a further refined proof was found that relied on n points in general position instead. We define the notion of general position below.

Definition 2.4 (General position)**.** $2k+d$ points on S^d are said to be in general position if no $d+2$ of the points lie on a hyperplane through the center of the sphere.

We first present this refined proof of the conjecture, which also generalizes easily to prove Dolnikov's theorem; then, we will present a second proof using Gale's lemma and a consequence known as Schrijver's theorem.

3. Proof of Kneser's conjecture

We begin by proving Kneser's conjecture, relying on the Lyusternik-Shnirel'man theorem as its main line of reasoning.

Proof. We start with our ground set of $2k+d$ points in general position on S^{d+1}. Assume that the vertex set V, which is made up of all k-sets of this ground set, can be partitioned into $d+1$ intersecting sets corresponding to color classes, as is needed for a $d+1$-coloring of a Kneser graph. Our job is to show that at least one of these classes contains 2 disjoint sets A, B, which would be a contradiction – color classes must be intersecting families in order for their vertices to not be connected. We set, for $i = 1, 2, \dots d+1$,

$$O_i = \big\{x \in S^{d+1} : \text{the open hemisphere}\, H_x \\ \text{with pole}\, x\, \text{contains a}\, k\text{-set from}\, V_i\big\}\,.$$

Each O_i, then, is an open set, and the open sets $O_1, O_2, \dots O_d$ plus the closed set

$$C = S^{d+1} \setminus \{O_1 \cup O_2 \cup \ldots \cup O_d\}$$

form a covering of S^{d+1}. By Lyusternik–Shnirel'man, one of these sets contains a pair of antipodal points $x^*, -x^*$. However, it cannot be C, because if it were, H_{x^*} and H_{-x^*} would contain less than k points, meaning that more than $d+2$ points would be on the equator formed by these hemispheres, which is a contradiction since the points are in general position. So, some O_i contains a pair of antipodal points, and there are k-sets A and B from the same color class in separate hemispheres H_{x^*} and H_{-x^*}. But because the hemispheres are open, they are disjoint, meaning A and B are disjoint, which completes the proof.

□

The proof of Kneser's conjecture is a pretty nice result about graph theory and finite sets that uses one of the staple results from topology. Now, we will see an extension to hypergraphs known as Dolnikov's theorem.

4. Dolnikov's theorem

Dolnikov's theorem deals with the coloring of Kneser hypergraphs, the notion of which comes naturally from the definition of ordinary Kneser graphs; its proof follows similar topological lines of reasoning as the proof of Kneser's conjecture. The content in this section comes from [2]. We begin by defining a hypergraph.

Definition 4.1. A *hypergraph*, or set system, is a collection of subsets of some ground set E, where each subset is called a hyperedge. For our purposes, we will use $[n]$ as our ground set.

Clearly, the Kneser graph on a set system $\mathcal{F}$ can be defined naturally: we have

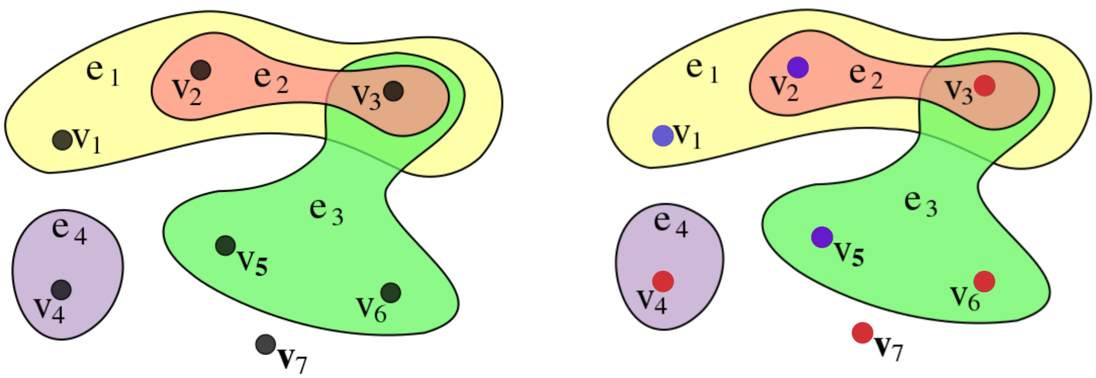
$$K(\mathcal{F}) = (\mathcal{F}, \{(A, B) : A, B \in \mathcal{F}, A \cap B = \emptyset\}),$$

ie, there is an edge between disjoint hyperedges (subsets).

Definition 4.2. A hypergraph is m-colorable if there is a mapping $E \to [m]$ such that each hyperedge contains vertices of at least two colors.

Definition 4.3. The *2-colorability defect* of a hypergraph $\mathcal{F}$, $cd_2(\mathcal{F})$, is the smallest number of vertices that must be removed to make the hypergraph 2-colorable.

Example. Consider the following hypergraph H, which can be 2-colored.



(a) H **(b)** A coloring

Figure 3. The hypergraph H and its coloring.

Theorem 4.4 (Dolnikov)**.** *For a hypergraph $\mathcal{F}$, we have*

$$\chi(K(\mathcal{F})) \geq cd_2(\mathcal{F}).$$

Equality does not necessarily hold, and there is no straightforward way to determine $cd_2(\mathcal{F})$. However, the proof of Dolnikov's theorem is similar to the proof of Kneser's conjecture.

Proof. Suppose $\chi(K(\mathcal{F})) = d$; again, take an arrangement of E, the ground set of $\mathcal{F}$, in general position on S^{d+1}. As in the previous proof, we define the sets O_i and C by

$$O_i = \{x \in S^{d+1} : \text{the open hemisphere } H_x \text{ with pole } x \text{ contains a } k\text{-set from } V_i\}.$$

and

$$C = S^{d+1} \setminus \{O_1 \cup O_2 \cup \ldots \cup O_d\}.$$

By the Lyusternik-Shnirel'man theorem, one of these $d+1$ sets contains a pair of antipodal points. Suppose this set is one of $O_1 \ldots O_d$. Then, there would be two k-sets from the same color class V_i in opposite, and thus, disjoint, hemispheres. Therefore, these two k-sets would be disjoint, which is a contradiction considering how we have constructed the color classes.

Therefore, C contains two antipodal points, and there is no set $A \in \mathcal{F}$ in either hemisphere H_{x^*} or H_{-x^*}. There are at most $\chi(K(\mathcal{F}))$ points on the equator; removing them and the sets in $\mathcal{F}$ containing them, we obtain a new hypergraph $\mathcal{F}'$, in which all sets touch both hemispheres H_{x^*} and H_{-x^*}. $\mathcal{F}$ can be colored by 2 colors corresponding to the hemispheres, so the chromatic number of $K(\mathcal{F})$ is at least the 2-colorability defect of $\mathcal{F}$. □

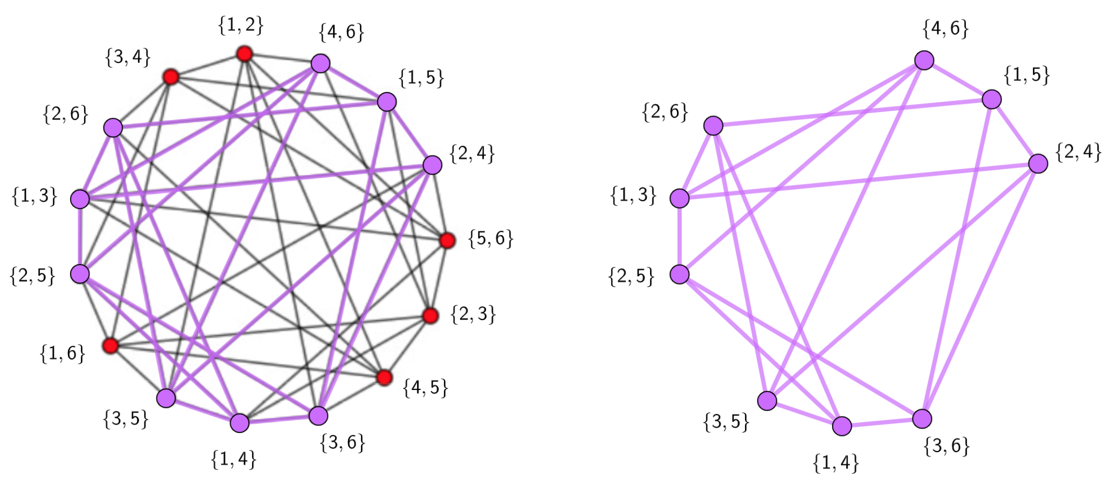
Dolnikov's theorem is significantly more powerful than Kneser's conjecture, as it applies to all hypergraphs, which include subsets of all possible sizes, rather than graphs of k-subsets. Next, we will look at a clever lemma, Gale's lemma, and use it to prove another related result.

5. Gale's lemma and Schrijver's theorem

Here, we present a separate proof of the Kneser conjecture based on Gale's lemma; this proof yields another extension of the theorem known as *Schrijver's theorem.* To motivate this, we first present Schrijver's theorem, which relies on the notion of a stable subset, and of the Schrijver graph. The material in this section comes from Jiri Matousek's *Using the Borsuk-Ulam Theorem* [3].

Definition 5.1 (Schrijver graph)**.** A subset S of $[n]$ is *stable* if it does not contain any neighboring elements modulo n (ie, if it contains the element i, it does not contain the element $i-1$ or $i+1$ modulo n). We denote the family of stable subsets by $[n]_{\text{stab}}$, and the Schrijver graph $S(n,k)$ is the Kneser graph on the set of stable k-subsets of $[n]$; in other words, an induced subgraph of $K(n,k)$.

Example. The following graph is the Schrijver graph $S(6,2)$, which is clearly an induced subgraph of $K(6,2)$.



(a) The induced Schrijver graph of $K(6,2)$

(b) $S(6,2)$

Theorem 5.2 (Schrijver's theorem)**.** $\chi(S(n,k)) = \chi(K(n,k)) = n - 2k + 2 = d + 2.$

Schrijver's theorem is quite interesting, as it states that even if as many as half of the vertices of a Kneser graph are removed, its chromatic number remains the same. In order to prove Schrijver's theorem, we will first state and prove Gale's lemma.

Theorem 5.3 (Gale's lemma)**.** *For $d \geq 0$ and $k \geq 1$, $2k+d$ points can be arranged on S^d so that each open hemisphere of S^d contains at least k of them.*

Proof. We will prove that there exist $2k+d$ points $v_1 \dots v_{2k+d}$ in $\mathbb{R}^{d+1}$ such that each open half-space (the open set produced by considering one half of the division of $\mathbb{R}^{d+1}$ by a hyperplane) whose boundary passes through the origin. Define the moment curve $\bar{\gamma}$ as

$$\bar{\gamma} = \{(1, t, t^2, \dots, t^d) \in \mathbb{R}^{d+1} : t \in \mathbb{R}\}.$$

We label any $2k+d$ points on $\bar{\gamma}$ as $w_1 \dots w_{2k+d}$ in order of their occurrence on $\bar{\gamma}$, and points are referred to as even or odd depending on their index. Also, we define $v_i = (-1)^i w_i$.

Let h be a hyperplane passing through the origin, and h^+ and h^- be the resultant open half-spaces. We have to show that for both h^+ and h^-, contain at least k points from the v_i's. For the half-space h^+, because $v_i = -w_i$ for odd i, and $v_i = w_i$ for even i, we have to show that the number of even w_i's in h^+ plus the number of odd w_i's in h^- is at least k.

Next, we claim that $\bar{\gamma}$ does not intersect h in more than d points, and that if it has d intersections with h, it crosses to opposite sides of h at each intersection. This has the following justification: if h has the equation $1 + a_1x_1 + a_2x_2 \dots + a_dx_d = b$, then if a point $\bar{\gamma}(t)$ lies in h, we have $1 + a_1t + a_2t^2 \dots + a_dt^d = b$, and we have a polynomial equation $p(t)$ of degree d in t, which obviously has at most d solutions. If there are indeed d distinct intersections, all roots of the $p(t)$ have multiplicity 1 and $p(t)$ changes sign at each root, meaning $\bar{\gamma}$ crosses between sides of h at each intersection.

We consider a hyperplane h through the origin, and move it continuously to contain d points of the set $W = \{w_1 \dots w_{2k+d}\}$, keeping the points of W on the same side of h throughout. This can be done using the following process: we start with some points of W, and to add more points, we rotate h around the origin and a $(d-2)$-flat (essentially a $(d-2)$-dimensional subset of $\mathbb{R}^{d+1}$) until h includes another w_i; repeat until h contains d points of W.

So, we can assume that h intersects $\bar{\gamma}$ in d points, all in W. We will define $W^+ = W \cap h$, and $W^- = W \setminus W^+$. Each point in W^+ represents a place where $\bar{\gamma}$ crosses from one side of h to the other.

We color each $w_i \in W^-$ black if it is even and lies in h^+ or if it is odd and lies in h^-; otherwise, it is colored white. Clearly, the black and white points of W^- alternate.
Take two consecutive points w and w' of W^-, and all the j points between them. If j is even, both w and w' are in the same half-space, and are of different colors, since one is even and the other is odd. If j is odd, then w, w' are in different half-spaces but are also of different colors. So, either way, there are at least $\lfloor \frac{|W^-|}{2} \rfloor \geq k$ black points, proving Gale's lemma. □

With Gale's lemma, we can present another proof of Kneser's conjecture, which is due to Lovász, that implies Schrijver's theorem.

Proof. Consider a Kneser graph $K(n,k)$, and $d = n - 2k$. Consider a set $X \subset S^d$ in the arrangement specified in Gale's lemma, where the points in X correspond to $[n]$.

Consider a $d+1$-coloring of $K(n,k)$ and define $O_1 \dots O_{d+1}$ in the usual fashion. Then, the O_i's are an open cover of S^d, since each open hemisphere contains at least one k-set (by Gale's lemma). By Lyusternik-Shnirel'man, one of these open hemispheres has a pair of

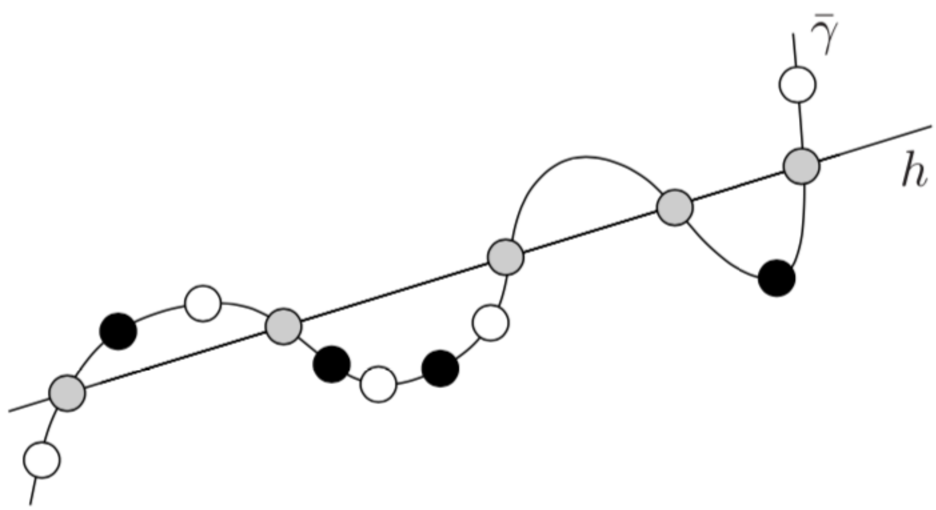


Figure 4. $\bar{\gamma}$ and the coloring of the w_i's.

antipodal points, and this leads to the same contradiction as before, because there are two k-sets of the same color in disjoint open hemispheres. □

This proof follows the same line of reasoning as the previous ones, but leads to Schrijver's theorem.

Theorem 5.4 (Schrijver's theorem). $\chi(S(n,k)) = \chi(K(n,k)) = n - 2k = d.$

Proof. The proof of Schrijver's theorem proceeds exactly as the second proof of Kneser's conjecture, with a slight modification regarding Gale's lemma: each open hemisphere should contain a stable k-set. Indeed, this is possible under our proof of Gale's lemma: a stable set is formed by the black points if $\bar{\gamma}$ is numbered according to $[n]$.

□

References

[1] Martin Aigner, Karl Heinrich Hofmann, and Ziegler Günter M. "The chromatic number of Kneser graphs". In: *Proofs from the Book*. Springer, 2010.

[2] Jacob Fox. *Lecture 14: Topological Methods*. URL: `math.mit.edu/~fox/MAT307-lecture14.pdf`.

[3] Jiri Matousek. *Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry*. Springer Publishing Company, Incorporated, 2007. ISBN: 3540003622.